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Utility representation via additive or multiplicative error functions ☆

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Abstract

The aim of this article is to study two distinct cases of utility representations where the error functions are assumed to display characteristics different than usual. These characteristics depend respectively on the feasible set and the two alternatives compared. Thus, in the first case the error functions are additive and in the second they are multiplicative.

Our study of additive error functions shows that a very narrow class of choice functions can be represented in this form. Also we introduce a new class of binary relations, called simple semiorders, to fill the relevant gap in the literature. As for the case of multiplicative error functions, we study the cases where the error function is directly and inversely proportional to the utility function. We show that these classes of binary relations display characteristics of interval orders, semiorder, or regular semiorders depending on the case studied.

Keywords: Utility function; Error function; Binary relation; Choice; Rationalizability

1. Introduction

The problem of describing human behavior through numerical representation, revealed preference and different conditions of rationality has become a popular area of

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research since the famous work by Samuelson [16] (more recent work includes that of Chernoff [10], Sen [17] and Suzumura [18]). Arrow [9] showed that choice is equivalent to the selection of undominated alternatives on some weak order according to the maximization of a utility function, and that the corresponding choice function satisfies the condition, which is often referred to as Arrow's choice Axiom.

Amstrong [6–8] drew attention to the fact that indifference relations are not transitive because the human mind is not necessarily capable of perfect discrimination, and introduced the notion of semiorders into economic theory. Yet semiorder was axiomatized in a more precise way by Luce [14] who introduced a new numerical representation for semiorders with a constant error. However, one limitation of these studies is that they all worked with constant errors. This idea was developed in many publications, the most recent of which are Fishburn [13] and Pirlot and Vincke [15]. Agaev and Aleskerov [3] and Aizerman and Aleskerov [4] take into consideration generalized models of interval choice in which the error functions are dependent on the feasible set of alternatives. Two cases of numerical representation with specifically defined error functions are analyzed in this paper. The first case of error functions depends on the set of feasible alternatives. This is an additive function that depends on the alternatives separately. The second case of error functions depends on compared alternatives x and y. We also assume that this function is multiplicative and we consider the cases when the error function $\varepsilon(x)$ depends on the utility function $u(x)$.

In Section 2, we give our results about representation of choice via a utility function and an additive error function that depends on the feasible set of alternatives. Section 3 contains results of the numerical representation of binary relations with a multiplicative error function depending on compared alternatives x and y. All proofs are given in Appendix A.

2. Choice representable via a utility function and an additive error function

Here we explore the finite set A of alternatives. A choice function is denoted as $C(\cdot)$ where the point in the brackets stands for some non-empty set $X \subseteq A$. As usual it is assumed $C(X) \subseteq X$ for any X. The utility function $u(\cdot)$ is a real-valued function defined on the set A. A binary relation P on a set A is a set of ordered pairs (x, y) with $x, y \in A$. We write xPy to mean that $(x, y) \in P$. Similarly, $x\bar{P}y$ means that $(x, y) \notin P$, i.e., (x, y) is not an element in P.

Definition 1. A choice function $C(\cdot)$ is said to have a numerical representation via a utility function with an error if there exist functions u and ε such that $\forall X \subseteq A$

$$C(X) = \{x \in X \mid \nexists\, y \in X \text{ s.t. } u(y) - u(x) > \varepsilon\}. \tag{1}$$

This means that there exists an insensitivity zone (or measurement error) ε in which these alternatives can be considered indifferent in terms of choice even if their utilities are different. For example, although the distinction between one and three cubes of sugar in a coffee makes a difference in taste, we would not be able to differentiate between the tastes of one and two cubes or between two and three cubes. In other words, we are indifferent between n and $n+1$ cubes, yet we have a definite preference between one and ten cubes.

Choice functions that have a numerical representation with error functions of the type $\varepsilon=\varepsilon(x,y,X)$, $\varepsilon=\varepsilon(y,X)$, and $\varepsilon=\varepsilon(X)$ were investigated by Agaev and Aleskerov [3], and Aizerman and Aleskerov [4]. In this section we build on this literature by studying a special case of the error function $\varepsilon=\varepsilon(X)$, namely, the case of an additive error function ε. The example below shows that there are choice functions which cannot be represented as in (1) with an error function of the form $\varepsilon=\varepsilon(X)$.

For the case of an error function of the form $\varepsilon=\varepsilon(X)$, (1) becomes $\forall X\subseteq A,\ X\neq\emptyset$,

$$C(X)=\{x\in X \mid \nexists\, y\in X \text{ s.t. } u(y)-u(x)>\varepsilon(X)\}. \tag{2}$$

Definition 2. A choice function $C(\cdot)$ is said to be rationalizable by a binary relation P if $\forall X\subseteq A,\ X\neq\emptyset$,

$$C(X)=\{x\in X \mid \nexists\, y\in X \text{ s.t. } yPx\}.$$

Example 1. There exists a binary relation P which rationalizes a choice function that cannot have a numerical representation as in (2). Let $A=\{a,b,c,d\}$ and consider the binary relation $P=\{(a,b),(c,d)\}$. Let $C(\cdot)$ be the choice function which is rationalizable by P. Consider the sets $\{a,b,d\}$ and $\{b,c,d\}$. It is seen that $C(\{a,b,d\})=\{a,d\}$ and $C(\{b,c,d\})=\{b,c\}$. Since d belongs to the choice set from $\{a,b,d\}$ but b does not, then

$$\exists x\in\{a,b,d\} \text{ s.t. } u(x)-u(b)>\varepsilon(\{a,b,d\})$$

and

$$\forall x\in\{a,b,d\},\quad u(x)-u(d)\leqslant\varepsilon(\{a,b,d\}).$$

These inequalities imply $u(b)<u(d)$. However $u(b)>u(d)$ is obtained when the set $\{b,c,d\}$ is considered. Thus, $C(X)$ cannot be represented as in (2).

Definition 3. A function $\varepsilon: 2^A\to\Re$ is said to be additive if $\forall X\subseteq A$

$$\varepsilon(X)=\sum_{x\in X}\varepsilon(x).$$

The additivity property states that the value of the error $\varepsilon(X)$ is the sum of the error values $\varepsilon(x)$ through X. If $\varepsilon(x)$ is assumed to be non-negative the error value of the set X increases along with its number of elements. It means that the insensitivity zone may increase by adding elements to the original choice set. As a result of this specification of the measurement error of the set X, an element that could not be chosen under the original set can now be selected under the expanded set.

Let us consider the representation of choice functions via a utility function and an additive error function. Then (2) becomes $\forall X \subseteq A$

$$C(X) = \left\{ x \in X \mid \nexists y \in X \text{ s.t. } u(y) - u(x) > \sum_{x \in X} \varepsilon(x) \right\}. \tag{3}$$

Below are the definitions of special types of binary relations (for detailed studies see e.g. [14,11,4, pp. 95–96]).

Definition 4. (a) A binary relation P is called a weak bi-order if for any distinct $x, y, z, w \in A$

$$xPy \wedge zPw \Rightarrow xPw \vee zPy \vee yPw \vee wPy \vee wPw \vee yPy$$

and

$$x_1 Px_2 Px_3 \ldots .x_r Px_1 \Rightarrow \exists i \in \{1, 2, \ldots, r\} \text{ (where } x_{r+1} = x_1) \\ x_{i+1} Px_i \vee x_i Px_i.$$

(b) A binary relation P is called a bi-order if $\forall x, y, z, w \in A$

$$xPy \wedge zPw \Rightarrow xPw \vee zPy.$$

(c) An irreflexive bi-order P is called an interval order.
(d) A bi-order which satisfies the condition $\forall x, y, z, w \in A$

$$xPyPz \Rightarrow xPw \vee wPz$$

is called a coherent bi-order.
(e) An irreflexive coherent bi-order P is called a semiorder.
(f) An irreflexive binary relation P is called a weak order if for any distinct $x, y, z \in A$

$$xPy \Rightarrow y\bar{P}x$$

and

$$xPy \Rightarrow xPz \vee zPy.$$

(g) A weak order P is called a linear order if for any distinct $x, y \in A$, $xPy \vee yPx$.

Let WBO, BO, IO, CBO, SO, WO and LO denote the set of the weak bi-orders, bi-orders, interval orders, coherent bi-orders, semiorders, weak orders and linear orders, respectively. It can easily be seen that CBO $\subset$ BO $\subset$ WBO, SO $\subset$ IO $\subset$ BO, SO $\subset$ CBO and LO $\subset$ WO.

Example 2. Let $A = \{a, b, c, d, e\}$ and consider the semiorder

$$P = \{(a, b), (a, c), (b, c), (a, d), (a, e)\}.$$

The choice function rationalizable by the semiorder P cannot be represented as in (3). To see this, we know that for all $x \in A$ $\varepsilon(x) \geqslant 0$ since P is irreflexive. Then we get $u(a) > u(b) > u(c)$. The next step is going to show that $u(b) > u(d)$, $u(e) > u(c)$.

Assume that $u(b) \leqslant u(d)$, then we have $\varepsilon(c) + \varepsilon(d) \geqslant u(d) - u(c) > \sum_{x \in \{b,c,d\}} \varepsilon(x)$ since $C(\{c,d\}) = \{c,d\}$ and $C(\{b,c,d\}) = \{b,d\}$. It gives us $0 > \varepsilon(b)$ which contradicts the fact that $\varepsilon(x) \geqslant 0$ for all $x \in A$. So we get $u(b) > u(d)$. To show $u(d) > u(c)$, assume that $u(d) \leqslant u(c)$. Then we get $u(b) - u(d) \geqslant u(b) - u(c) > \sum_{x \in \{b,c,d\}} \varepsilon(x)$ since $c \notin C(\{b,c,d\})$ and $u(b) > u(d)$. This contradicts $C(\{b,c,d\}) = \{b,d\}$. Since $c \notin C(\{b,c,d,e\})$ and $C(\{b,d\}) = \{b,d\}$, $(\sum_{x \in \{b,d\}} \varepsilon(x)) + u(d) \geqslant u(b) > (\sum_{x \in A/\{a\}} \varepsilon(x)) + u(c)$. Then by $C(\{c,d\}) = \{c,d\}$, we have $\varepsilon(c) + \varepsilon(d) + u(c) \geqslant u(d) > \varepsilon(c) + \varepsilon(e) + u(c)$. Therefore, we get $\varepsilon(d) > \varepsilon(e)$. When we replace d with e, we will have $\varepsilon(d) < \varepsilon(e)$ which contradicts our previous result.

The example has shown that there exists a choice function rationalizable by a semiorder which cannot be represented as in (3). However, it can easily be seen that a choice function which is rationalizable by a weak order can be represented as in (3). Thus, the question arises: Is it possible to find a class of binary relations which is a proper subset of the semiorders such that the choice functions rationalizable via these binary relations can be represented as in (3)?

Before we state our next result, let us construct the partitions which define the structure of an interval order (see, e.g. [12]).

The strong intervality condition ($\forall x, y, z, w \in A xPy \wedge zPw \Rightarrow xPw \vee zPy$) implies that $\forall x, y$, $L(x) \subseteq L(y)$ *or* $L(y) \subseteq L(x)$, where $L(x)$ is the lower contour set of x in P, i.e., $L(x) = \{y \in A \,|\, xPy\}$. Irreflexivity indicates that there is a chain with respect to the lower contour sets, i.e., $L(x_1) \subset L(x_2) \ldots L(x_{n-1}) \subset L(x_n)$.

Let us construct the sets

$$I_k = \{x \in A \mid L(x_k) = L(x)\},$$

where $k = 1, \ldots, n$ (n is finite by the finiteness of A). I_k is not empty for any k since $x_k \in I_k$ by construction. Clearly, the system $\{I_k\}_1^n$ is a partition of the set A, i.e., $\bigcup_{k=1}^n I_k = A$, $I_k \cap I_l = \emptyset$ when $k \neq l$. Now construct another family of non-empty sets $\{J_m\}_1^n$, as follows:

$$J_1 = L(x_2) \setminus L(x_1),$$

$$J_2 = L(x_3) \setminus L(x_2),$$

$$\vdots$$

$$J_{n-1} = L(x_n) \setminus L(x_{n-1}),$$

$$J_n = A \Bigg\backslash \bigcup_{m=1}^{n-1} J_m \,.$$

Clearly, the system $\{J_m\}_1^n$ is a partition of the set A, i.e., $\bigcup_{m=1}^n J_m = A$, $J_k \cap J_m = \emptyset$ when $k \neq m$. Then any interval order P can be represented as

$$P = \bigcup_{k=2}^{n} \left[I_k \times \bigcup_{m=1}^{k-1} J_m \right].$$

We now introduce a special type of semiorders by restricting the number of elements in the partitions $I_k \cap J_m$.

Definition 5. A semiorder will be called a simple one if

$$|I_k \cap J_m| \leqslant 1 \quad \text{if } |k - m| \leqslant 1,$$

$$|I_k \cap J_m| = 0 \quad \text{otherwise},$$

where $\{I_k\}_1^n$ and $\{J_m\}_1^n$ are the partitions of A.

"Simplicity" implies that not only $I_k \cap J_m = \emptyset$ for $k \leqslant m - 2$ but also $I_k \cap J_k$ and $I_k \cap J_{k+1}$ are at most singletons; this also implies that the cardinality of I_k is at most 2. Moreover, in a simple semiorder, at most three elements can be indifferent. On Figure 1, two semiorders are shown: a simple one, and one that is not simple. The "rectangles" represent I_k while the "circles" represent J_m, separating $I_k \cap J_k$ from $I_k \cap J_{k+1}$.

It is worth mentioning here that simple semiorders are a generalization of linear orders: when $I_k = J_k$ for all k, a simple semiorder turns out to be a linear order. However, in contrast to semiorders, simple semiorders are not a generalization of weak orders since the indifference classes of a simple semiorder are singletons.

Theorem 6. *A choice function which is rationalizable by a simple semiorder P can be represented as* (3).

Example 3. Let $A = \{a, b, c\}$ and the utilities and the errors be as in the following table:

	$u(\cdot)$	$\varepsilon(\cdot)$
a	2	0
b	1	-1
c	0	2

It can easily be seen that there is no binary relation P that rationalizes the choice function representable in the form of (3) with the utility and error functions given above. Therefore, the choice function representable in the form (3) is not rationalizable by any binary relation. Hence, the inverse statement of Theorem 6 is not true.

The main idea of the theorem is to find a class of binary relations such that the choice functions rationalizable by these binary relations can be represented with the error function $\varepsilon(X)$ that is the sum of the error values $\varepsilon(x)$ through X. In other words, $\varepsilon(X)$ may change as the number of elements which belong to X is altered. When an individual has a lot of alternatives to choose from, it will be difficult to make a decision among them. The representation with a constant error cannot capture these kind of situations. Also we showed that there are some choice functions that are rationalizable by a semiorder P that cannot be represented as in (3). Is it possible, then, to find a class of

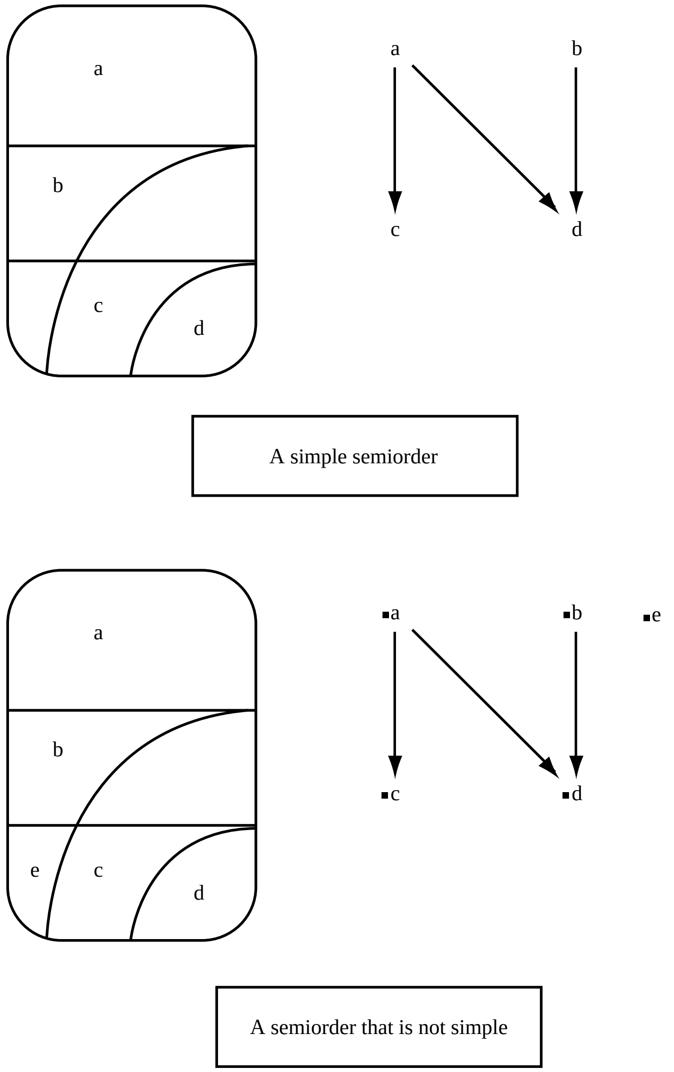

Fig. 1.

binary relations which is a proper superset of the simple semiorders such that the choice functions rationalizable by these binary relations can be represented as in (3)? We still do not know the answer to this question. Theorem 6 only shows that the representation in the form of (3), an additive error function, arises when the corresponding binary relation is a simple semiorder.

To conclude this section, we construct a generalization of the interval orders to explore if we can find a class of binary relation represented by (3).

Definition 7. An interval order will be called a simple one if

$$|I_k \cap J_m| \leqslant 1,$$

where $\{I_k\}_1^n$ and $\{J_k\}_1^n$ are the partitions of A.

Remark 1. A choice function which is rationalizable by a simple interval order P cannot be represented as in (3). Let $A=\{a,b,c,d,e\}$, and consider the simple interval order $\check{P}=\{(a,b),(a,c),(b,c),(a,e)\}$. The choice function rationalizable by the interval order $\check{P}$ cannot be represented as in (3).

To see this, we know that $\varepsilon(x) \geqslant 0$ for all $x \in A$ since P is irreflexive. And we have $\varepsilon(a)+\varepsilon(c)+2\varepsilon(d) \geqslant u(a)-u(c) > \sum_{x\in A}\varepsilon(x)$ since $C(\{a,d\})=\{a,d\}$, $C(\{c,d\})=\{c,d\}$ and $c \notin C(A)$. It gives us $\varepsilon(d) > \varepsilon(b)+\varepsilon(e)$. At the same time, we have $\varepsilon(b)+\varepsilon(c)+2\varepsilon(e) \geqslant u(b)-u(c) > \sum_{x\in\{b,c,d,e\}}\varepsilon(x)$ since $C(\{b,e\})=\{b,e\}$, $C(\{c,e\})=\{c,e\}$, and $c \notin C(\{b,c,d,e\})$. It gives us $\varepsilon(d) < \varepsilon(e)$ which contradicts the fact that $0 \leqslant \varepsilon(b)$.

3. Numerical representation of binary relations with multiplicative error functions

We begin this section by providing the definition of a numerical representation of a binary relation via a utility function with an error. Then we discuss briefly the existing results in the literature.

Definition 8. A binary relation P is said to have a numerical representation via utility function with an error if there exist two functions $u(\cdot)$ and ε such that

$$xPy \;\Leftrightarrow\; u(x)-u(y) > \varepsilon. \tag{4}$$

Depending on the form of the function ε one can obtain different types of binary relations P in (4). If ε=constant $\geqslant 0$, then P is a semiorder [14]. If $\varepsilon=\varepsilon(x) \geqslant 0$, then P is an interval order [12]. If we omit the restriction for ε to be non-negative, then one can obtain more general classes of relations—coherent bi-orders and bi-orders, respectively (see [11]). Moreover, any of the above-mentioned relations can be represented as in (4) with appropriate functions u and ε.

In Aleskerov and Vol'skiy [5], Agaev and Aleskerov [3], Abbas and Vincke [2], Abbas [1], and Aizerman and Aleskerov [4] the model studied is one in which ε in (4) depends on both comparable alternatives x and y. It has been shown that if ε is non-negative, then any non-cyclical binary relation can be represented in this way, and if ε is not restricted, then any binary relation has such numerical representation. In the case where the error function depends additively on $\varepsilon(x)$ and $\varepsilon(y)$, i.e.,

$$\varepsilon_1(x,y)=\varepsilon(x)+\varepsilon(y),$$

the corresponding P is an interval order (see [4]).

Now we are interested in the case where the error function is multiplicative, i.e.,

$$\varepsilon_1(x,y)=\varepsilon(x)\varepsilon(y)$$

and we consider the function ε to be dependent in a different way on the value of the utility function u.

We will consider two cases: one, in which the error value decreases when the utility value increases; this corresponds to the case when alternatives with small utility values are considered to be similar. The second case is where the error value increases along with the utility values; this corresponds to the case when alternatives with high utilities are considered as similar. We can exemplify these two cases in the following way: in the first case, an affluent man does not feel the need to differentiate between the prices in the supermarket and the local bazaar, the latter being comparatively cheap. On the contrary, in the second case, a poor man would not to be able to distinguish between two luxury cars because of their inaccessibility.

Theorem 9. *Let P have a numerical representation with an error such as that in* (4), *the utility function $u(\cdot)$ be positive, the error function $\varepsilon_1(\cdot,\cdot)$ be multiplicative, i.e.*, $\forall x,y\ \varepsilon_1(x,y)=\varepsilon(x)\varepsilon(y)$, *and the function $\varepsilon(x)$ depend on $u(x)$ in such a way that $\varepsilon(x)=\alpha/u(x)$ with $\alpha>0$. Then P is an interval order.*

Theorem 9 shows the conditions for P to be an interval order. However, proving the inverse statement remains an open problem.

Example 4. An interval order P with a numerical representation as stated in Theorem 9 is not necessarily a semiorder. This can be shown by the following example: let $u(x)=9,\ u(y)=7,\ u(z)=4,\ u(w)=1$ and $\alpha^2=80$. Then it is easily seen that $xPyPz$ but $x\bar{P}w$ and $w\bar{P}z$.

Theorem 10. *Let P have a numerical representation with an error such as that in* (4), *the utility function $u(\cdot)$ be positive, the error function $\varepsilon_1(\cdot,\cdot)$ be multiplicative, i.e.*, $\forall x,y\ \varepsilon_1(x,y)=\varepsilon(x)\varepsilon(y)$, *and the function $\varepsilon(x)$ depend on $u(x)$ in such a way that $\varepsilon(x)=\alpha u(x)$ with $\alpha>0$. Then P is a semiorder.*

The inverse statement of Theorem 10 has been proved only for a special sub-class of the class of semiorders referred to as regular semiorders. Before introducing those relations we first recapitulate interval orders (hence, semiorders), which use two partitions of the set A.

Any semiorder P can be represented as

$$P=\bigcup_{k=2}^{n}\left[I_k\times\bigcup_{m=1}^{k-1}J_m\right],$$

where $\{I_k\}_1^n$ and $\{J_m\}_1^n$ are the partitions of A (see Section 2).

Now we can introduce the regular semiorders.

Definition 11. A semiorder will be called "regular" if

$$I_k \cap J_m = \emptyset \quad \text{if } k \leqslant m - 2,$$

where $\{I_k\}_1^n$ and $\{J_m\}_1^n$ are the partitions of A.

The following lemma describes a category of regular semiorders.

Lemma. *Let P be a semiorder with the property $I_k \cap J_m \neq \emptyset$ for any $k = 1, \ldots, m$. Then $I_k \cap J_k = \emptyset$ for any $k \leqslant m - 2$.*

Proof. Let $u \in I_m \cap J_m$, $v \in I_{m-1} \cap J_{m-1}$ and $w \in I_{m-2} \cap J_{m-2}$. It is easy to show that $uPvPw$. Suppose there is $x \in I_{m-2} \cap J_m$; since $x \in J_m$, we have $u\bar{P}x$; similarly, from $x \in I_{m-2}$, we get $x\bar{P}w$. This contradicts the fact that P is a semiorder. Supposing there is $x \in I_k \cap J_m$ for $k < m - 2$ leads to the same contradiction. □

For this type of semiorders the inverse statement of Theorem 10 turns out to be true. However, this remains an open question for a semiorder.

Theorem 12. *Any regular semiorder P has a numerical representation with an error such as that in* (4), *the utility function $u(\cdot)$ being positive, the error function $\varepsilon_1(\cdot,\cdot)$ being multiplicative, i.e., $\forall x, y \ \varepsilon_1(x, y) = \varepsilon(x)\varepsilon(y)$, and the function $\varepsilon(x)$ depends on $u(x)$ in such a way that $\varepsilon(x) = \alpha u(x)$ with $\alpha > 0$.*

Example 5. Let $A = \{a, b, c, d\}$ and consider the interval order $P = \{(a, b), (b, d), (a, d)\}$. This interval order cannot have a numerical representation with an error such as that in (4) where the utility function $u(\cdot)$ is positive, the error function $\varepsilon_1(\cdot,\cdot)$ is multiplicative, i.e., $\forall x, y \ \varepsilon_1(x, y) = \varepsilon(x)\varepsilon(y)$, and the function $\varepsilon(x)$ depends on $u(x)$ such that $\varepsilon(x) = \alpha u(x)$ with $\alpha > 0$. Since P is an irreflexive binary relation, without loss of generality we can assume that the utility value of each alternative is non-negative. $bPd \wedge c\bar{P}d \Leftrightarrow u(b) - u(d) > \alpha^2 u(b)u(d)$ and $u(c) - u(d) \leqslant \alpha^2 u(c)u(d) \Leftrightarrow (u(b) - u(d))/\alpha^2 u(b) > u(d) \geqslant (u(c) - u(d))/\alpha^2 u(c) \Leftrightarrow u(b) > u(c)$. This implies that $u(a) - u(b) < u(a) - u(c)$. We have $\alpha^2 u(a)u(b) < \alpha^2 u(a)u(c)$ since $aPb \wedge a\bar{P}c$. So we find $u(b) < u(c)$ which contradicts the previous result.

Remark 2. It can easily be shown that any weak order admits the representation in (4) with a multiplicative error function and $\varepsilon(x) = \alpha u(x)$ or $\varepsilon(x) = \alpha/u(x)$. Indeed, let us consider the error function of the form $\varepsilon(x) = \alpha u(x)$. The weak order P is defined by the partition $\{Z_m\}_1^n$ such that

$$xPy \text{ iff } x \in Z_i, y \in Z_j, \quad \text{and} \quad i > j.$$

If we choose α to be equal to $1/n$, and $u(x)$ to be equal to i if $x \in Z_i$, then for two elements in different classes Z_i and Z_j, ε will be less than 1, and for two elements from the same class, ε will be equal to 1.

For the second type of error function one can choose α to be equal to 1. It can be shown that such a function ε satisfies the necessary requirement.

Appendix A

Proof of Theorem 6. For each $x \in I_k \cap J_m$ set the value of u to

$$u(x) = \left\{ \begin{array}{ll} 2^{k+1} & \text{if } k = m \\ 3 \cdot 2^k & \text{otherwise} \end{array} \right\}$$

and the error function ε to

$$\varepsilon(x) = \left\{ \begin{array}{ll} 0 & \text{if } k = m, \\ 2^k & \text{otherwise} \end{array} \right\}.$$

Consider a choice function $C(X)$ rationalizable by a simple semiorder P. Denote by $\hat{C}(\cdot)$, the choice function that is represented in (3) with $u(\cdot)$ and $\varepsilon(\cdot)$ as defined above.

Let us prove that $\hat{C}(X) = C(X)$ for all $X \subset A$.

If $x \in C(X)$, then $\nexists y \in X$ s.t. yPx.

Let $x \in I_k \cap J_m$. Since P is simple, $m = k$ or $m = k + 1$.

i) If $x \in I_k \cap J_{k+1}$, then $u(x) = 3 \cdot 2^k$. If $u(y) - u(x) < 0$, we are done. So we need to consider two cases to show that $\forall y \in X, \quad u(y) - u(x) \le \sum_{z \in X} \varepsilon(z)$. If $y \in I_{k+1} \cap J_{k+1}$, then $u(y) - u(x) = 2^{k+2} - 3 \cdot 2^k = 2^k = \varepsilon(x) \le \sum_{z \in X} \varepsilon(z)$. If $y \in I_{k+1} \cap J_{k+2}$, then $u(y) - u(x) = 3 \cdot 2^{k+1} - 3 \cdot 2^k = 3 \cdot 2^k = \varepsilon(x) + \varepsilon(y) \le \sum_{z \in X} \varepsilon(z)$.

ii) If $x \in I_k \cap J_k$, then $u(x) = 2^{k+1}$. The maximal value for $u(y)$ with $y \in X \subseteq \bigcup_{m=1}^{k} J_m$ is eventually attained when $y \in I_k \cap J_{k+1}$ since then $u(y) = 3 \cdot 2^k$. In such a case $u(y) - u(x) = 3 \cdot 2^k - 2^{k+1} = 2^k = \varepsilon(y) \le \sum_{z \in X} \varepsilon(z)$. In all other cases $u(y) - u(x) < 0 \le \sum_{z \in X} \varepsilon(z)$. Therefore $x \in \hat{C}(X)$.

If $x \in X$ and $x \notin C(X)$, then $\exists y \in X$ s.t. yPx; if there are several such y's take the one for which $u(y)$ is maximal. Now let us prove that $u(y) - u(x) > \sum_{z \in X} \varepsilon(z)$.

Since P is a simple semiorder, $y \in I_k \cap J_m$ with $m = k$ or $m = k + 1$. yPx implies $x \in I_{k-j}, j \ge 1$ and $x \in J_{m-l}, l \ge 1$. Moreover, $y \in I_k$ and yPx imply x cannot belong to J_k since $J_k = L(x_{k+1}) \setminus L(x_k)$. Hence, if $y \in I_k \cap J_{k+1}$, $x \notin I_{k-1} \cap J_k$.

i) If $y \in I_k \cap J_k$, then $u(y) = 2^{k+1}$; the worst case for x is $x \in I_{k-1} \cap J_{k-1}$, hence $u(x) = 2^k$. We then have: $u(y) - u(x) = 2^{k+1} - 2^k = 2^k$ and $\varepsilon(X) = \sum_{z \in X} \varepsilon(z) \le 2^1 + 0 + 2^2 + \dots\dots + 2^{k-1} + \underbrace{\varepsilon(y)}_{0} = 2^k - 2 < 2^k$.

ii) If $y \in I_k \cap J_{k+1}$, then $u(y) = 3 \cdot 2^k$. Since $x \notin I_{k-1} \cap J_k$, the worst case we have to consider is $x \in I_{k-1} \cap J_{k-1}$, hence $u(x) = 2^k$. We then have: $u(y) - u(x) = 3 \cdot 2^k - 2^k = 2^{k+1}$ and $\varepsilon(X) = \sum_{z \in X} \varepsilon(z) \le 2^1 + 0 + 2^2 + \dots\dots + 2^{k-1} + \underbrace{\varepsilon(y)}_{2^k} = 2^{k+1} - 2 < 2^{k+1}$. Therefore, in both cases $x \notin \hat{C}(X)$.

Consequently, for all x in X and $\forall X \in 2^A$ $\hat{C}(X) = C(X)$. $\square$

Proof of Theorem 9. (i) P is irreflexive. Since $\varepsilon_1(x,x)=\varepsilon(x)\varepsilon(x)=(\alpha/u(x))(\alpha/u(x))=\alpha^2/u^2(x)>0$, then $\varepsilon_1(x,x)>0=u(x)-u(x)$. Thus $x\bar{P}x$.

(ii) P satisfies strong intervality. Assume on the contrary that $xPy\wedge zPw\wedge x\bar{P}w\wedge z\bar{P}y$. Then

$$u(x)-u(y)>\frac{\alpha}{u(x)}\frac{\alpha}{u(y)}, \tag{A.1}$$

$$u(z)-u(w)>\frac{\alpha}{u(z)}\frac{\alpha}{u(w)}, \tag{A.2}$$

$$u(x)-u(w)\leqslant\frac{\alpha}{u(x)}\frac{\alpha}{u(w)}, \tag{A.3}$$

$$u(z)-u(y)\leqslant\frac{\alpha}{u(z)}\frac{\alpha}{u(y)}. \tag{A.4}$$

Adding (A.1)–(A.4) we obtain

$$\frac{\alpha}{u(x)}\frac{\alpha}{u(y)}+\frac{\alpha}{u(z)}\frac{\alpha}{u(w)}<\frac{\alpha}{u(z)}\frac{\alpha}{u(y)}+\frac{\alpha}{u(x)}\frac{\alpha}{u(w)}.$$

Multiplying both sides by $u(x)u(y)u(z)u(w)/\alpha^2$, we obtain

$$u(x)u(y)+u(z)u(w)<u(x)u(w)+u(y)u(z).$$

Then $u(z)u(w)-u(y)u(z)<u(x)u(w)-u(x)u(y)$, so

$$u(z)(u(w)-u(y))<u(x)(u(w)-u(y)). \tag{A.5}$$

Moreover, (A.1) and (A.3) imply

$$u^2(x)u(y)-u(x)u^2(y)>\alpha^2\geqslant u^2(x)u(w)-u(x)u^2(w), \tag{A.6}$$

(A.2) and (A.4) imply

$$u^2(z)u(w)-u(z)u^2(w)>\alpha^2\geqslant u^2(z)u(y)-u(z)u^2(y). \tag{A.7}$$

Consider three possible cases in (A.5)

1. $u(w)>u(y)$,
2. $u(w)<u(y)$,
3. $u(w)=u(y)$.

Case 1: $u(w)>u(y)\Rightarrow u(x)>u(z)$.

From (A.6) $u^2(x)(u(w)-u(y))<u(x)(u^2(w)-u^2(y))$, which implies

$$u^2(x)(u(w)-u(y))<u(x)(u(w)-u(y))(u(w)+u(y)),$$

then

$$u(x)<u(w)+u(y). \tag{A.8}$$

From (A.7) $u^2(z)(u(w)-u(y))>u(z)(u^2(w)-u^2(y))$, which implies $u^2(z)(u(w)-u(y))>u(z)(u(w)-u(y))(u(w)+u(y))$, then

$$u(z)>u(w)+u(y). \tag{A.9}$$

From (A.8) and (A.9) we get $u(z) > u(x)$ which contradicts $u(z) < u(x)$ obtained before in (A.5).

Case 2: $u(w) < u(y) \Rightarrow u(x) < u(z)$.

From (A.6) $u^2(x)(u(w) - u(y)) < u(x)(u^2(w) - u^2(y))$, which implies

$$u(x) > u(w) + u(y). \tag{A.10}$$

From (A.7) $u^2(z)(u(w) - u(y)) > u(z)(u^2(w) - u^2(y))$, which implies

$$u(z) < u(w) + u(y). \tag{A.11}$$

From (A.10) and (A.11) we get $u(z) < u(x)$ which contradicts $u(z) > u(x)$.

Case 3: $u(w) = u(y)$.

From (A.6) $u^2(x)u(y) - u(x)u^2(y) > \alpha^2 \geqslant u^2(x)u(y) - u(x)u^2(y)$ which contradicts $\alpha > 0$. $\square$

Proof of Theorem 10. (i) P is irreflexive. Since $\varepsilon_1(x,x) = \varepsilon(x)\varepsilon(x) = \alpha^2 u^2(x) > 0$, then $\varepsilon_1(x,x) > 0 = u(x) - u(x)$. Thus $x\bar{P}x$.

(ii) P satisfies strong intervality. Assume on the contrary that $xPy \wedge zPw \wedge x\bar{P}w \wedge z\bar{P}y$. Then

$$u(x) - u(y) > \alpha^2 u(x)u(y), \tag{A.12}$$

$$u(z) - u(w) > \alpha^2 u(z)u(w), \tag{A.13}$$

$$u(x) - u(w) \leqslant \alpha^2 u(x)u(w), \tag{A.14}$$

$$u(z) - u(y) \leqslant \alpha^2 u(z)u(y), \tag{A.15}$$

(A.12) and (A.14) imply that

$$u(y) < \frac{u(x)}{1 + \alpha^2 u(x)} \leqslant u(w) \Rightarrow u(y) < u(w),$$

(A.13) and (A.15) imply that

$$u(w) < \frac{u(z)}{1 + \alpha^2 u(z)} \leqslant u(y) \Rightarrow u(w) < u(y),$$

a contradiction.

(iii) P is semitransitive. Assume on the contrary that $xPy \wedge yPz \wedge x\bar{P}w \wedge w\bar{P}z$. Then

$$u(x) - u(y) > \alpha^2 u(x)u(y), \tag{A.16}$$

$$u(y) - u(z) > \alpha^2 u(y)u(z), \tag{A.17}$$

$$u(x) - u(w) \leqslant \alpha^2 u(x)u(w), \tag{A.18}$$

$$u(w) - u(z) \leqslant \alpha^2 u(w)u(z), \tag{A.19}$$

(A.16) and (A.18) imply that

$$u(y) < \frac{u(x)}{1+\alpha^2 u(x)} \leqslant u(w).$$

Hence $u(y) < u(w)$; add $\alpha^2 u(w)u(y)$ to both sides,

$$\alpha^2 u(w)u(y) + u(y) < u(w) + \alpha^2 u(w)u(y),$$

from this $(\alpha^2 u(w)+1)u(y) < u(w)(1+\alpha^2 u(y))$, we obtain

$$\frac{u(y)}{1+\alpha^2 u(y)} < \frac{u(w)}{1+\alpha^2 u(w)}.$$

At the same time, (A.17) and (A.19) imply that

$$\frac{u(y)}{1+\alpha^2 u(y)} > \frac{u(w)}{1+\alpha^2 u(w)},$$

i.e., we obtain a contradiction. □

Proof of Theorem 12. Any semiorder P can be represented as

$$P = \bigcup_{k=2}^{n} \left[I_k \times \bigcup_{m=1}^{k-1} J_m \right].$$

Now define $\{Z_m\}_2^{2n}$ s.t.

$$Z_m = \begin{cases} I_{m/2} \cap J_{m/2} & \text{if } m \text{ is even,} \\ I_{(m-1)/2} \cap J_{(m+1)/2} & \text{otherwise,} \end{cases}$$

since $I_k \cap J_m = \emptyset$ for all $|k-m| > 1$. The sets Z_m are pairwise disjoint and their union is the set A, i.e., $\bigcup_{m=2}^{2n} Z_m = A$, $Z_k \cap Z_l = \emptyset$ when $k \neq l$. Moreover, it is possible that Z_m is empty for some m.

Now construct the numerical function $u(\cdot)$,

$\forall x, y \in Z_m$ $u(x) = u(y)$ and denoted it by $u(x_m)$, and define another function $\psi(\cdot)$ s.t. $\psi(x) = (u(x)/1 - \alpha^2 u(x))$ for all $x \in A$. The functions $u(\cdot)$ and $\psi(\cdot)$ are determined recursively starting from $u(x_{2n})$ assuming $\alpha^2 = 1/(n+1)$.

$$u(x_{2n}) = n, \quad \psi(x_{2n-1}) = \frac{u(x_{2n}) + \psi(x_{2n})}{2}$$

and

$$\psi(x_k) = \begin{cases} \dfrac{u(x_{k+1}) + u(x_{k+2})}{2} & \text{if } k \text{ is even,} \\ \dfrac{\psi(x_{k+1}) + u(x_{k+2})}{2} & \text{otherwise,} \end{cases} \quad \text{for all } 2 \leqslant k \leqslant 2n-2.$$

We would like to note that $\psi(x_{2n}) = u(x_{2n})/1 - \alpha^2 u(x_{2n}) = n/1 - (1/(n+1))n = n(n+1) > u(x_{2n})$.

Claim. *For all* $k < 2n, u(x_k) < u(x_{2n})$.

Proof. First, let us prove $u(x_{2n-1})$ is less than $u(x_{2n})$. Assume $u(x_{2n-1}) \geqslant u(x_{2n})$, in other words $\psi(x_{2n-1})/(1+\alpha^2\psi(x_{2n-1})) \geqslant \psi(x_{2n})/(1+\alpha^2\psi(x_{2n}))$. It implies that $\psi(x_{2n-1})+\alpha^2\psi(x_{2n-1})\psi(x_{2n}) \geqslant \psi(x_{2n})+\alpha^2\psi(x_{2n-1})\psi(x_{2n})$. We would have $\psi(x_{2n-1}) \geqslant \psi(x_{2n})$ in contradiction with the definition of $\psi(x_{2n-1})$ and the fact that $u(x_{2n}) < \psi(x_{2n})$. Hence $u(x_{2n-1}) < u(x_{2n})$.

$\psi(x_{2n-2})=(u(x_{2n-1})+u(x_{2n}))/2$ implies $u(x_{2n-1}) < \psi(x_{2n-2}) < u(x_{2n})$ since $u(x_{2n-1}) < u(x_{2n})$. It can easily be seen that $\psi(x_{2n-2}) < u(x_{2n}) < \psi(x_{2n-1})$. Assume $u(x_{2n-2}) \geqslant u(x_{2n})$; put differently $\psi(x_{2n-2})/(1+\alpha^2\psi(x_{2n-2})) \geqslant \psi(x_{2n})/(1+\alpha^2\psi(x_{2n}))$. It implies that $\psi(x_{2n-2}) \geqslant \psi(x_{2n})$. It is contradiction. So $u(x_{2n-2}) < u(x_{2n})$.

Assume $u(x_t) < u(x_{2n})$ for all $2n-2 > t > k-1$ and show $u(x_{2n}) > u(x_{k-1})$. Assume $u(x_{2n}) \leqslant u(x_{k-1})$. For the case when k is even, $\psi(x_k) = (u(x_{k+1}) + u(x_{k+2}))/2$ implies $\psi(x_k) < u(x_{2n}) < \psi(x_{2n})$. Meanwhile, $u(x_{2n}) \leqslant u(x_{k-1})$ gives us $\psi(x_{2n}) \leqslant \psi(x_k)$ that contradicts the previous result. Hence $u(x_k) < u(x_{2n})$. For odd k, $\psi(x_k) = (\psi(x_{k+1}) + u(x_{k+2}))/2$ and $\psi(x_{k+1}) < u(x_{2n})$ imply that $\psi(x_k) < u(x_{2n}) < \psi(x_{2n})$. By the same reasoning explained above, $u(x_k) < u(x_{2n})$. Therefore, $u(x_k) < u(x_{2n})$ for all $2 \leqslant k \leqslant 2n$. This proves the claim. We would like to stress the fact that $u(x_k)$ and $\psi(x_k)$ are clearly positive by construction.

The claim shows that $1-\alpha^2u(x_k)$ is always greater than zero for all $2 \leqslant k \leqslant 2n$.

Let us derive two inequalities which will be useful later.

$$u(x_k) < u(x_l) \quad \Leftrightarrow \quad \psi(x_k) < \psi(x_l) \quad \text{for } 2 \leqslant k,\ l \leqslant 2n. \tag{A.20}$$

Indeed, $\psi(x_k) < \psi(x_l) \Leftrightarrow \dfrac{u(x_k)}{1-\alpha^2u(x_k)} < \dfrac{u(x_l)}{1-\alpha^2u(x_l)}$

$$\Leftrightarrow u(x_k) - \alpha^2u(x_l)u(x_k) < u(x_l) - \alpha^2u(x_l)u(x_k)$$

$$\Leftrightarrow u(x_k) < u(x_l).$$

Another inequality is

$$\psi(x_m) < u(x_k) < \psi(x_l) \quad \Leftrightarrow \quad x_kPx_m \wedge x_k\bar{P}x_l \tag{A.21}$$

Indeed, $\psi(x_m) < u(x_k) < \psi(x_l)$

$$\Leftrightarrow \frac{u(x_m)}{1-\alpha^2u(x_m)} < u(x_k) < \frac{u(x_l)}{1-\alpha^2u(x_l)}$$

$$\Leftrightarrow u(x_m) < u(x_k) - \alpha^2u(x_m)u(x_k) \quad \text{and} \quad u(x_k) - \alpha^2u(x_l)u(x_k) < u(x_l)$$

$$\Leftrightarrow u(x_k) - u(x_m) > \alpha^2u(x_k)u(x_m) \quad \text{and} \quad u(x_k) - u(x_l) < \alpha^2u(x_k)u(x_l)$$

$$\Leftrightarrow u(x_k) - u(x_m) > \varepsilon_1(x_k,x_m) \quad \text{and} \quad u(x_k) - u(x_l) > \varepsilon_1(x_k,x_l)$$

$$\Leftrightarrow x_kPx_m \wedge x_k\bar{P}x_l. \qquad \square$$

Lemma. *If* $2 \leqslant k < l \leqslant 2n$ *then* $u(x_k) < u(x_l)$.

Proof. $\psi(x_{2n-1}) = (u(x_{2n}) + \psi(x_{2n}))/2$ and $u(x_{2n}) < \psi(x_{2n})$ imply $\psi(x_{2n-1}) < \psi(x_{2n})$. By (A.20), $u(x_{2n-1}) < u(x_{2n})$.

$\psi(x_{2n-2}) = (u(x_{2n-1}) + u(x_{2n}))/2$ implies $u(x_{2n-1}) < \psi(x_{2n-2}) < u(x_{2n})$ since $u(x_{2n-1}) < u(x_{2n})$. It can easily be seen that $\psi(x_{2n-2}) < u(x_{2n}) < \psi(x_{2n-1})$. By (A.20) $u(x_{2n-2}) < u(x_{2n-1})$.

Assume that k is fixed and $u(x_m) < u(x_{m+1})$ for $k < m \leqslant 2n-1$. We need to prove that $u(x_k) < u(x_{k+1})$.

For odd, k, $\psi(x_{k+1}) = (u(x_{k+2}) + u(x_{k+3}))/2$ and $u(x_{k+2}) < u(x_{k+3})$ imply that $u(x_{k+2}) < \psi(x_{k+1})$. It gives us that $u(x_{k+2}) < \psi(x_k) < \psi(x_{k+1})$ by construction of $\psi(x_k)$. Hence $u(x_k) < u(x_{k+1})$ by (A.20).

For even k, $u(x_{k+1}) < u(x_{k+2})$ implies $\psi(x_{k+1}) < \psi(x_{k+2})$. It gives us $u(x_{k+3}) < \psi(x_{k+1})$ since $\psi(x_{k+1}) = (\psi(x_{k+2}) + u(x_{k+3}))/2$. By $\psi(x_k) = (u(x_{k+1}) + u(x_{k+2}))/2$, $u(x_{k+1}) < \psi(x_k) < u(x_{k+2}) < u(x_{k+3}) < \psi(x_{k+1})$ since $u(x_{k+1}) < u(x_{k+2}) < u(x_{k+3})$. Then we have $\psi(x_k) < \psi(x_{k+1})$. By (A.20), $u(x_k) < u(x_{k+1})$. Therefore, we can say that $u(x_k) < u(x_l)$ for all $2 \leqslant k < l \leqslant 2n$. This proves the statement of the lemma.

Now, let us prove the theorem. It can easily be shown that ψ is decreasing with respect to k. Now, we need to prove that the binary relation P has a numerical representation via the utility function which was defined above. Assume that $k > 3$ is even, then we have

$$\psi(x_k) = \frac{u(x_{k+1}) + u(x_{k+2})}{2}, \tag{A.22}$$

$$\psi(x_{k-1}) = \frac{\psi(x_k) + u(x_{k+1})}{2}. \tag{A.23}$$

$\psi(x_{k-1}) < \psi(x_k)$ and (A.23) imply that $u(x_{k+1}) < \psi(x_{k-1})$. And $u(x_{k+1}) < u(x_{k+2})$ and (A.22) imply that $u(x_{k+1}) < \psi(x_k) < u(x_{k+2})$. Hence

$$u(x_k) < u(x_{k+1}) < \psi(x_{k-1}) < \psi(x_k) < u(x_{k+2}) \tag{A.24}$$

since $u(x_k)$ is increasing with respect to k. Then $\psi(x_{k-2}) = (u(x_{k-1}) + u(x_k))/2$ and (A.24) imply that

$$\psi(x_{k-2}) < u(x_k) < u(x_{k+1}) < \psi(x_{k-1}) < \psi(x_k). \tag{A.25}$$

(A.25) and (A.21) imply that x_k cannot beat x_{k-1}. Then x_k cannot beat x_m for $2n \geqslant m \geqslant k-1$ since $\psi(x_k)$ is increasing with respect to k. We can also say that x_k beats x_{k-2}. Then x_k beats x_m for $2 \leqslant m \leqslant k-2$ since $\psi(x_k)$ is increasing. They also imply that x_{k+1} cannot beat x_m for $2n \geqslant m > k-2$ and x_k beats x_m for $2 \leqslant m \leqslant k-2$. All these cases exhaust the proof. □

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